Probabilistic Methods in Combinatorics

Instructor: Oliver Janzer

Assignment 6

To solve for the Example class on 1st April. Submit the solution of Problem 1 by Sunday 30th March if you wish feedback on it. Some hints will be given on Friday 28th March.

The solution of each problem should be no longer than one page!

Problem 1. Let $1/3 \le p_1, p_2, \ldots, p_n \le 1$ be reals. For every $i \in [n]$, let X_i be a Bernoulli random variable of parameter p_i, such that $X_1, \ldots, X_n$ are independent. Show that

$$\mathbb{P}\left(\left|\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} p_i\right| \ge 10\sqrt{\sum_{i=1}^{n} p_i}\right) \le 1/100.$$

Problem 2. Let X be a random variable taking nonnegative integer values. In the lectures we have seen that $\mathbb{P}(X=0) \le \frac{\mathrm{Var}(X)}{\mathbb{E}[X]^2}$. Prove that in fact

$$\mathbb{P}(X=0) \le \frac{\mathrm{Var}(X)}{\mathbb{E}[X^2]}.$$

Problem 3. Let $G=(V,E)$ be a simple graph with n vertices and m edges, and let k be a positive integer. Prove that:

(a) There are at least $k^n \cdot (1 - \frac{m}{k})$ proper vertex-colourings of G with k colours.

(b) There are at most $k^n \cdot \frac{k-1}{m}$ proper vertex-colourings of G with k colours.

(c) The upper bound from (b) can be improved to $k^n \cdot \frac{k-1}{k+m-1}$.

Problem 4. Let $v_1 = (x_1, y_1), \ldots, v_n = (x_n, y_n)$ be n two-dimensional vectors, where each x_i and each y_i is an integer whose absolute value does not exceed $2^{n/2}/(100\sqrt{n})$. Show that there are two disjoint sets $I, J \subseteq \{1, 2, \ldots, n\}$ such that

$$\sum_{i \in I} v_i = \sum_{j \in J} v_j.$$